CAPM: The theory

Lecture 4

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April 23, 2014

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Section 1

Notation

Notation

R_E	random return on an efficient portfolio
$\mu_E = \mathrm{E}[R_E]$	expected return on an efficient portfolio
R_M	random return on market portfolio of risky assets
$\mu_M = \mathrm{E}[R_M]$	expected return on market porfolio of risky assets
σ_E	standard deviation of R_E
σ_M	standard deviation of R_M

Section 2

Scope of the model

Scope of the model

Markowitz' portfolio theory describes behaviour of individual investors.

- Portfolio theory selects a portfolio, given the expected returns and covariances.
- Portfolio theory ('mean-variance analysis') is relevant for each investor, regardless of whether the CAPM is correct or not.

Scope of the model

CAPM is an equilibrium model specifying a relation between expected rates of return and covariances for all assets.

In capital market equilibrium, no investor wants to buy or to sell.

Scope of the model

Definition of the problem

If everyone in the economy holds an efficient portfolio, how will securities be priced in equilibrium?

Scope of the model

Contributors

William Sharpe of Stanford received the Nobel Prize in 1990 for his contribution, John Lintner of Harvard died before the prize was awarded.

Scope of the model

Assumptions

1. No transaction costs.
2. Assets are all tradable and are all infinitely divisible.
3. No taxes.
4. No individual can effect security prices (perfect competition).
5. Investors care only about expected returns and variances.
6. Unlimited short sales and borrowing and lending.
7. Homogeneous expectations.

Scope of the model

Assumptions

Assumptions 5 – 7 imply:

- All investors behave according to Markowitz' Portfolio Theory.
- All investors see the same efficient frontier.

Section 3

Mutual fund theorem

Mutual fund theorem

Mutual fund theorem

- In capital market equilibrium, all investors hold the same portfolio of risky assets, the tangency portfolio.
- Therefore the tangency portfolio equals the market portfolio or risky assets.

Mutual fund theorem

Definition ('market portfolio of risky assets')

A portfolio of all risky securities held in proportion to their market value. This must be the sum over all securities, i.e. stocks, bonds, real-estate, human capital, etc.

Mutual fund theorem

Interpreting market equilibrium

- In market equilibrium, every investor must be content with their portfolio holdings, i.e. nobody wants to buy or to sell.
- Leverage differs by investor. In market equilibrium, borrowing and lending at the riskless rate has to level out.
- In market disequilibrium, prices of securities have to change.

Section 4

Equilibrium returns

Equilibrium returns

Definition (Capital Market Line)

The Capital Market Line provides the set of efficient combinations of the market porfolio of risky assets and the riskless asset in market equilibrium:

$$E[R_E] = r_0 + \left(\frac{E[R_M] - r_0}{\sigma_M} \right) \cdot \sigma_E$$

Equilibrium returns

Definition (Security Market Line)

A linear relationship between the expecated value of an asset in market equilibrium and its beta:

$$\mu_i(\beta_i) = r_0 + \beta_i \cdot (\mu_M - r_0)$$

with

$$\beta_i = \frac{COV(R_i, R_M)}{(\sigma_M)^2}$$

Equilibrium returns

Interpretation of the Security Market Line (SML)

$(\mu_i - r_0)$ equilibrium risk premium on asset i

$(\mu_M - r_0)$ equilibrium risk premium on market porfolio of risky assets

Equilibrium returns

Interpretation of the Security Market Line (SML)

In case of disequilibrium of capital market:

$\mu_i > r_0 + \beta_i \cdot (\mu_M - r_0)$: asset i is undervalued

$\mu_i < r_0 + \beta_i \cdot (\mu_M - r_0)$: asset i is overvalued

Equilibrium returns

Interpretation of the Security Market Line (SML)

- Riskless asset: Beta is zero.
  - Return not correlated with return on market porfolio.
  - Return has to equal the riskless interest rate.
- Market porfolio of risky assets: Beta is one.

Section 5

Trade-off between risk and return

Trade-off between risk and return

Definition (Sharpe Ratio)

The Sharpe Ratio is the relation of the risk premimum and the risk:

$$S_i = \frac{\mu_i - r_0}{\sigma_i}$$

Trade-off between risk and return

The Sharpe Ratio of the market porfolio of risky assets is the slope of the Capital Market Line.

$$\lambda = \frac{\mu_M - r_0}{\sigma_M}$$

Trade-off between risk and return

The Sharpe Ratio of the market porfolio of risky assets (λ) can be interpreted as the equilibrium return/risk trade-off.

- All market participants are holding the market portfolio, which is also the tangency portfolio.
- The tangency portfolio portfolio has by definition the highest Sharpe Ratio of all portfolios.

Trade-off between risk and return

Comparison of equilibrium risk premimums

Conversion of SML:

$$\mu_i - r_0 = \frac{\mathrm{COV}(R_i, R_M)}{\mathrm{VAR}(R_M)} \cdot (\mu_M - r_0)$$

$$\Rightarrow \quad \frac{\mu_1 - r_0}{\mathrm{COV}(R_1, R_M)} = \ldots = \frac{\mu_m - r_0}{\mathrm{COV}(R_m, R_M)}$$

Trade-off between risk and return

Risk of market porfolio of risky assets

$$
\begin{aligned}
\sigma_M^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \cdot x_j \cdot \mathrm{COV}(R_i, R_j) \\
&= \sum_{i=1}^{n} x_i \cdot \left[\sum_{j=1}^{n} \cdot x_j \cdot \mathrm{COV}(R_i, R_j) \right] \\
&= \sum_{i=1}^{n} x_i \cdot \mathrm{COV}[R_i, R_M]
\end{aligned}
$$